Euler and Hamiltonian Path

- **Euler Path**
  a path visits every edge exactly once

- **Hamiltonian Path**
  a path visits every vertex exactly once

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**Euler Path**

- **Seven Bridges of Königsberg**
  (Traveling Salesperson Problem)
  A walk through the city that would cross each of bridges once
### Euler Path

- **Euler Path**: a path visits every edge exactly once

- **Euler Cycle**: Euler path which starts and stops at the same vertex

- A connected graph $G$ is called **Eulerian** if it contains an Euler path

#### Observation from an Euler path,
- $a > c > a > d > e > a > b$
- $a > c > b > a > d > f > a$

#### Theorem 1
A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree

#### Theorem 2
A connected multigraph has Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree
Euler Path

Euler Path

How to identify an Euler Path / Cycle?
- Euler Path
- Euler Cycle

Fleury’s Algorithm

Hierholzer’s Algorithm

Identify Euler Path

1. If there are 0 odd vertices, start anywhere. If there are 2 odd vertices, start at one of them.
2. Follow edges one at a time. If you have a choice between a bridge and a non-bridge, always choose the non-bridge.
3. Stop when you run out of edges.

“Don’t burn bridges” so that we can come back to a vertex and traverse remaining edges.
Hierholzer’s Algorithm

- Identify Euler Cycle
  1. Select a node \( v \) as a starting node
  2. Form a cycle using non-traveled edges and end at \( v \) (remove the visited edges)
  3. While all edges have been traversed, stop
     a) Find a node \( u \) on the previous cycles that’s connected to a non-traveled edge
     b) Form a cycle using non-traveled edges and end at \( u \) (remove the visited edges)
     c) Merge both tours at the node \( u \)

Example

- Is it possible to begin in a room or the outside and take a walk that goes through each door exactly once? If yes, how?

Example

1. Select a node \( v \) as a starting node
2. Form a cycle using non-traveled edges and end at \( v \) (remove the visited edges)
3. While all edges have been traversed, stop
   a) Find a node \( v \) on the previous cycles that’s connected to a non-traveled edge
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   c) Merge both tours at the node \( v \)
**Hamiltonian Path**

- **Icosian game**
  - Invented by an Irishman named Sir William Rowan Hamilton (1805-1865)
  - Is there a cycle in the dodecahedron puzzle that passes through each vertex exactly once?

**Dodecahedron puzzle**

**Hamilton Path**

- **Hamilton Path**: a path visits every vertex exactly once
- **Hamilton Cycle**: Hamilton path which starts and stops at the same vertex
- Self-loop and multiple edges can be ignored

**Dirac’s Theorem**

- **Theorem**: If each vertex of a simple graph with $n$ vertices and $n \geq 3$ has degree $\geq n/2$, there is a Hamilton circuit

**Ore’s Theorem**

- **Theorem**: If every pair of non-adjacent vertices $u$ and $v$ in a simple graph with $n$ vertices and $n \geq 3$ has $\text{deg}(u) + \text{deg}(v) \geq n$, there is a Hamilton circuit
Dirac’s and Ore’s Theorem

- Be noted Dirac’s and Ore’s Theorem is a sufficient condition but not necessary one
  - A graph with a vertex degree < n/2 may have a Hamilton circuit
  - A graph with a pair of non-adjacent vertices \( \text{deg}(u) + \text{deg}(v) < n \) may have a Hamilton circuit

Unfortunately, no good algorithm to find the Hamilton path or cycle

Just “trial and error” (and good luck!)

Euler Path VS Hamilton Path

- **Euler Path**
  - a path uses every edge exactly once
- **Euler Cycle**
  - Euler path which starts and stops at the same vertex
- **Hamilton Path**
  - a path uses every vertex exactly once
- **Hamilton Cycle**
  - Hamilton path which starts and stops at the same vertex

Planar Graph

- **Planar Graph** is a graph *can be drawn* in the plane without edges crossing
- A planar graph *drawn* in the plane without edges crossing is called **Plane Graph**
  - Plane graph is also called a planar representation of the graph

Isomorphism

Planar Graph

- Planar Graph
  - Planar Graph
  - Planar Graph
  - Planar Graph
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  - Planar Graph
  - Planar Graph
  - Planar Graph
Planar Graph

- A graph that is drawn in the plane is also said to be embedded (or imbedded) in the plane
- A planar graph can generate different plane graphs
- Application: Circuit Layout Problems

Planar Graph: Region

- A plane graph splits the plane into regions
  - Including the unbounded (exterior) region

Planar Graph: Region

- The vertices and edges of G that are incident with a region R form a subgraph of G called the boundary of R

Planar Graph: Region

- Observation on boundary
  - Cycle edge belongs to the boundary of two regions
  - Bridge is on the boundary of only one region (unbounded region)

\[ a > b > c > e > a \] is a cycle
\[ (a,b), (b,c), (a,e) \] belongs to R₁ and R₄
\[ (c,e) \] belongs to R₁ and R₂
\[ (e,b) \] is not a cycle, just a bridge
\[ (e,b) \] belongs to R₄ only
Planar Graph

- Is $K_{3,3}$ a planar graph? Not planar

1) Focus on a, e, d, b
2) c connects to d, e, f
3) f connects to a, b, c

Planar Graph

Euler’s Formula

- If $G$ be a connected planar simple graph with $e$ edges, $v$ vertices, and $r$ regions, then $r = e - v + 2$

- MI is used in the proof

Euler’s Formula: Proof

- For a connected planar graph $G$
  - Let a sequence of subgraphs $G_1$, $G_2$, $\ldots$,$G_i$, $\ldots$, $G_e$ of $G$, and $G_e = G$,
    - $G_1 \subset G_2 \subset \ldots \subset G_e$
    - $G_i$ contains $i$ edges
    - $G_n$ is obtained from $G_{n-1}$ by arbitrarily adding an edge
  - Be noted that all $G_i$ are planar (as subgraph of planar graph must be planar)

- Assume $r_n = e_n - v_n + 2$ is true
Planar Graph

Euler’s Formula: Proof

- Let \( \{a_{n+1}, b_{n+1}\} \) be the edge that is added to \( G_n \) to obtain \( G_{n+1} \)
  - Case 1: \( a_{n+1}, b_{n+1} \) are in \( G_n \)
  - Case 2: one of \( a_{n+1} \) and \( b_{n+1} \) is not in \( G_n \)

**Case 1:** \( a_{n+1}, b_{n+1} \) are in \( G_n \)

- \( e_{n+1} = e_n + 1 \), and \( v_{n+1} = v_n \)
- If \( a_{n+1} \) and \( b_{n+1} \) are not on the boundary of a common region \( R \), two edges cross. This violates \( G_{n+1} \) is planar
- Therefore, \( a_{n+1} \) and \( b_{n+1} \) must be on the boundary of a common region \( R \)
- The new edge splits \( R \) into two regions
  - \( r_{n+1} = r_n + 1 \)
  - Given \( r_n = e_n - v_n + 2 \)
  - \( (r_{n+1} - 1) = (e_{n+1} - 1) - (v_{n+1} - 1) + 2 \)
  - \( r_{n+1} = e_{n+1} - v_{n+1} + 2 \)

**Case 2:** one of \( a_{n+1} \) and \( b_{n+1} \) is not in \( G_n \)

- \( e_{n+1} = e_n + 1 \)
- \( v_{n+1} = v_n + 1 \)
- No new region is generated, \( r_{n+1} = r_n \)
- Given \( r_n = e_n - v_n + 2 \)
- \( (r_{n+1} - 1) = (e_{n+1} - 1) - (v_{n+1} - 1) + 2 \)
- \( r_{n+1} = e_{n+1} - v_{n+1} + 2 \)

Planar Graph

**Euler’s Formula: Example**

- Suppose that a connected planar simple graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph slit the plane?
  - \( v = 20 \)
  - Sum of degree = \( 20 \times 3 = 60 = 2e \)
  - \( e = 30 \)
  - \( r = e - v + 2 = 30 - 20 + 2 = 12 \)
Planar Graph

Euler’s Formula: Corollary

- If a connected planar simple graph, then G has a vertex of degree not exceeding 5.

- If a connected planar simple graph has e edges and v vertices with v ≥ 3, then e ≤ 3v – 6.

- If a connected planar simple graph has e edges and v vertices with v ≥ 3 and no circuits of length three, then e ≤ 2v – 4.

Planar Graph

Euler’s Formula: Example 1

- Show that K₅ is nonplanar.

- K₅ has circuit of length three, 5 vertices and 10 edges.

- As e = 10 and 3v – 6 = 9, e ≤ 3v – 6 is false.

- Therefore, K₅ is nonplanar.

Planar Graph

Euler’s Formula: Example 2

- Show that K₃,₃ is nonplanar.

- K₃,₃ has no circuit of length three, 6 vertices and 9 edges.

- As e = 9 and 2v – 4 = 8, e ≤ 2v – 4 is false.

- Therefore, K₃,₃ is nonplanar.

Planar Graph

Homeomorphic

- The graphs are called homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivision.

- If a graph is planar, it will be any graph obtained by removing an edge {u,v} and adding a new vertex w with edges {u,w} and {w,v}.

- Obtain G from H
  - Remove {a, b}, Add {a, e}, {e, b}
  - Remove {a, c}, Add {a, f}, {f, c}
  - Remove {f, c}, Add {f, c}, {c, g}
**Planar Graph**

**Kuratowski’s Theorem**

- A graph is not planar if it contains a non-planar subgraph
- **Kuratowski’s Theorem**
  A graph is nonplanar if it contains a subgraph homeomorphic to $K_{3,3}$ or $K_5$
- Proof is neglected

**Example**

- Determine whether the following graph is planar
  - As $G$ contains a subgraph $(H)$ homeomorphic to $K_5$, it is not planar

**Coloring**

- Two regions sharing a border are assigned different colors
- Represent a map by a graph (called **Dual Graph**)
  - Vertex: Region
  - Edge: Constraint
    - the color cannot be the same for adjacent regions

**Map Coloring**

- What is the largest complete graph represented by a map?
**Coloring**

- **Graph Coloring Problem**
  Given a graph, assign all the vertices with the minimum number of colors so that no two adjacent vertices gets the same color.

  ![Graph Coloring Example](image)

- **Chromatic number** ($\chi(G)$)
  The smallest number of colors needed to produce a proper coloring of $G$.

**Coloring: Example**

- **Cycle Graph (C)**
  - $\chi(C_{\text{even}}) = 2$
  - $\chi(C_{\text{odd}}) = 3$

- **Wheel Graph (W)**
  - $\chi(W_{\text{even}}) = 3$
  - $\chi(W_{\text{odd}}) = 4$

- **Complete Graph (K)**
  - $\chi(K_{\text{even}}) = n$
  - $\chi(K_{\text{odd}}) = n$

- **Tree (T)**
  - $\chi(T) = 2$
Coloring: Example

- Bipartite Graph
  - Recall... a graph is bipartite if all vertices can be partitioned into two partitions, so that any two adjacent vertices are in different partitions
  - Obviously, \( \chi = 2 \)

![Bipartite Graph Example](image)

Coloring

- No formula for Chromatic number \( \chi \)

Discussion

- Given a graph of size \( k \)
  - \( \chi > k \): not possible
  - \( \chi = k \): for a complete graph
  - \( \chi < k \): other graphs except the complete one

- Analyzing a subgraph of a graph may be helpful
  - If a subgraph is complete of size \( k \), \( \chi \geq k \)

\[ \chi = 4 \]

Coloring: Application 1

- A flight need a gate in an airport
- How many gates needed for this flight schedule? 3

Vertex: Flight
Edge: Share the same time slot

![Flight Application Diagram](image)

Coloring: Application 2

- Examination of subject conflicts if student takes both subjects
- How many different time slots? 3

Vertex: Course
Edge: a student take the two courses

![Course Application Diagram](image)